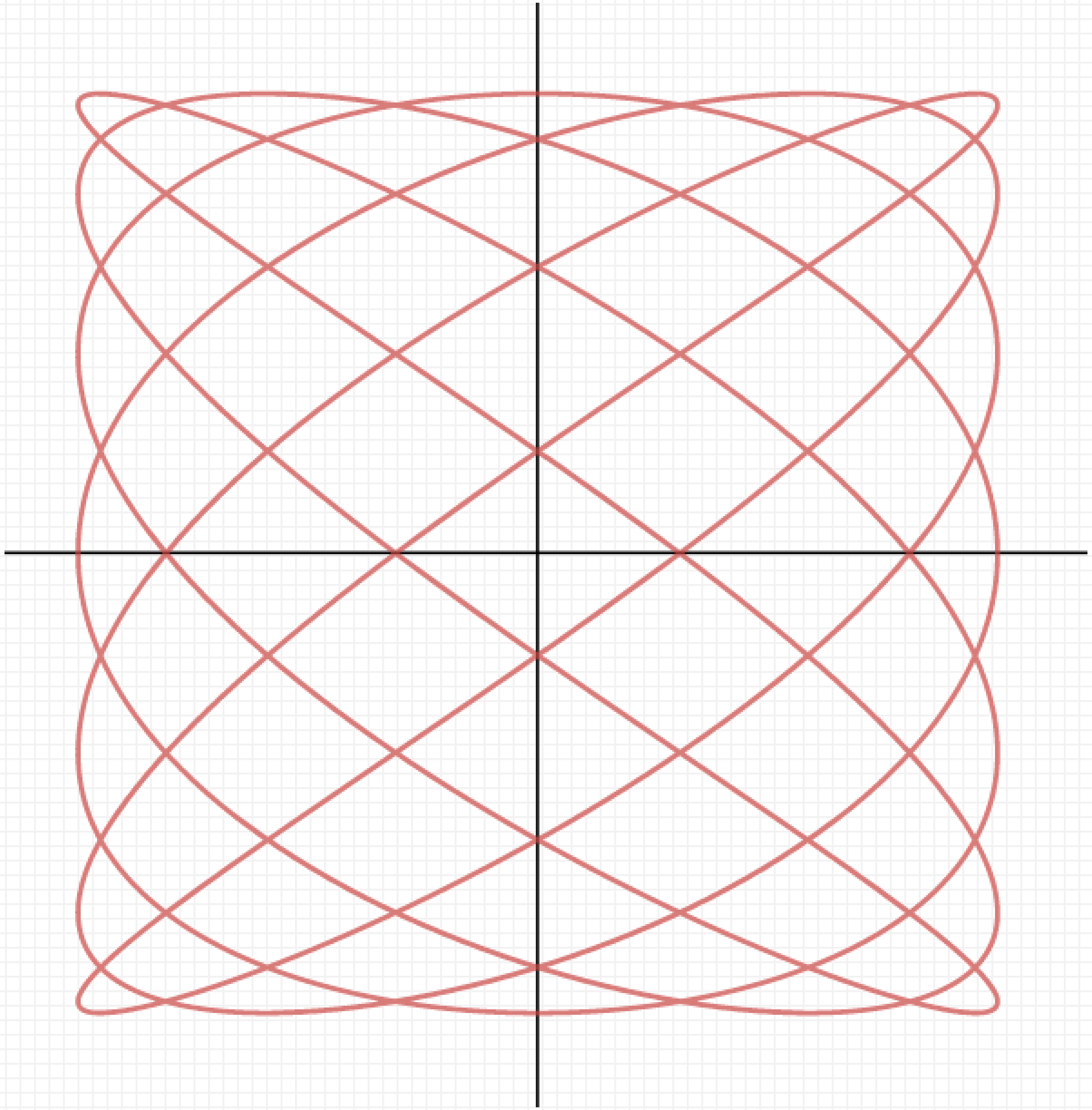
Section 10.1

Calculus of Parametric Functions



Greg Kelly, Hanford High School, Richland, Washington

Derivative at a point – parametrically: a parametric curve $x = x(t)$, $y = y(t)$ has a derivative at t_0 if $x(t)$ and $y(t)$ have a derivative at t_0.

The curve is differentiable if it is differentiable at all values of t.

The curve is smooth if $x'(t)$ and $y'(t)$ are continuous and not simultaneously zero.

The formula for finding the slope of a parametric curve is:

$$\frac{dy}{dx} = \frac{d/dt}{dx/dt}(y) = \frac{\frac{d}{dt}(y)}{\frac{dx}{dt}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{y'(t)}{x'(t)}$$

This makes sense if we think about cancelling *dt*.

We assume that the denominator is not zero.

Find the equation of the line tangent to $x = t^2$, $y = t^3$ at (1,1).

FIND THE t THAT GIVES YOU (1,1).

$\therefore t = 1$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2}{2t} = \frac{3}{2}t \Rightarrow \left.\frac{dy}{dx}\right|_{t=1} = \frac{3}{2} = m_{\text{TAN}}$$

$$\therefore \boxed{(y-1) = \frac{3}{2}(x-1)} \Rightarrow y = \frac{3}{2}x - \frac{1}{2}$$

Check: Convert to Cartesian coordinates, find dy/dx

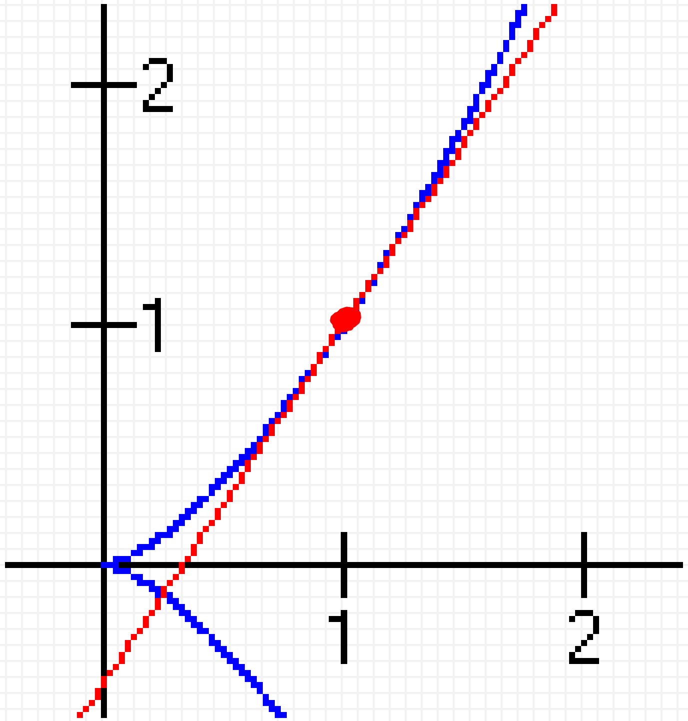
$$x^{1/2} = y^{1/3} \Rightarrow x^3 = y^2 \Rightarrow 3x^2 = 2y\frac{dy}{dx}$$

$\rightarrow$

$$3x^2 = 2y\frac{dy}{dx}$$

$$\frac{3x^2}{2y} = \frac{dy}{dx}$$

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{3(1)^2}{2(1)} = \frac{3}{2}$$



→

To find the <u>second</u> derivative of a parametric curve, we find the derivative of the first derivative:

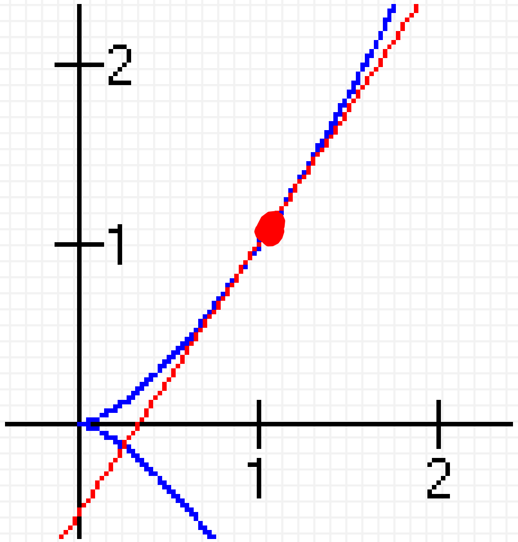
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

1. Find the first derivative (*dy/dx*).
2. Find the derivative of *dy/dx* with respect to *t*. NUMERATOR
3. Divide by *dx/dt*. DENOMINATOR

→

Find the second derivative for $x = t^2$, $y = t^3$ at (1,1) $\rightarrow t=1$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\overset{\frac{3}{2}t}{\overbrace{\frac{dy}{dx}}}\right)}{\frac{dx}{dt}} = \frac{\frac{3}{2}}{2t} = \frac{3}{4t} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=1} = \left.\frac{3}{4t}\right|_{t=1} \boxed{= \frac{3}{4}} > 0$$



∴ GRAPH SHOULD BE CC↑ AT (1,1).

→

Use the parametric equations to the right to answer the following questions.

$$x = t - t^2$$
$$y = t - t^3$$

1. Find the equation of the tangent line when t = 2.

POINT : $(x(2), y(2)) = (-2, -6)$

SLOPE : $\left.\frac{dy}{dx}\right|_{t=2} = \left.\frac{y'(t)}{x'(t)}\right|_{t=2} = \left.\frac{1-3t^2}{1-2t}\right|_{t=2} = \frac{-11}{-3}$

$$\boxed{y+6 = \frac{11}{3}(x+2)}$$

2. Find the second derivative and determine the concavity of the curve when t = 2.

$$\frac{d^2y}{dx^2} = \frac{\dfrac{(1-2t)(-6t)-(1-3t^2)(-2)}{(1-2t)^{\cancel{2}\,3}}}{\cancel{(1-2t)}} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{(1-4)(-12)-(-11)(-2)}{(1-4)^3} = \frac{14}{-27} < 0$$

$\therefore$ CURVE IS CONCAVE DOWN AT $t=2$.

→

Use the parametric equations to the right to answer the following questions.

$$x = t - t^2$$

$$y = t - t^3$$

3. Find all value of *t* for which the graph has horizontal and vertical tangents.

HORIZONTAL: $\frac{dy}{dx} = 0 \Rightarrow y'(t) = 0$

$(x'(t) \neq 0)$

$y' = 1 - 3t^2 = 0$

$3t^2 = 1$

$t^2 = 1/3$

$\boxed{t = \pm\sqrt{1/3}}$

VERTICAL TANGENTS: $\frac{dy}{dx}$ IS UND $\Rightarrow x'(t) = 0$

$(y'(t) \neq 0)$

$x' = 1 - 2t = 0$

$\boxed{t = 1/2}$

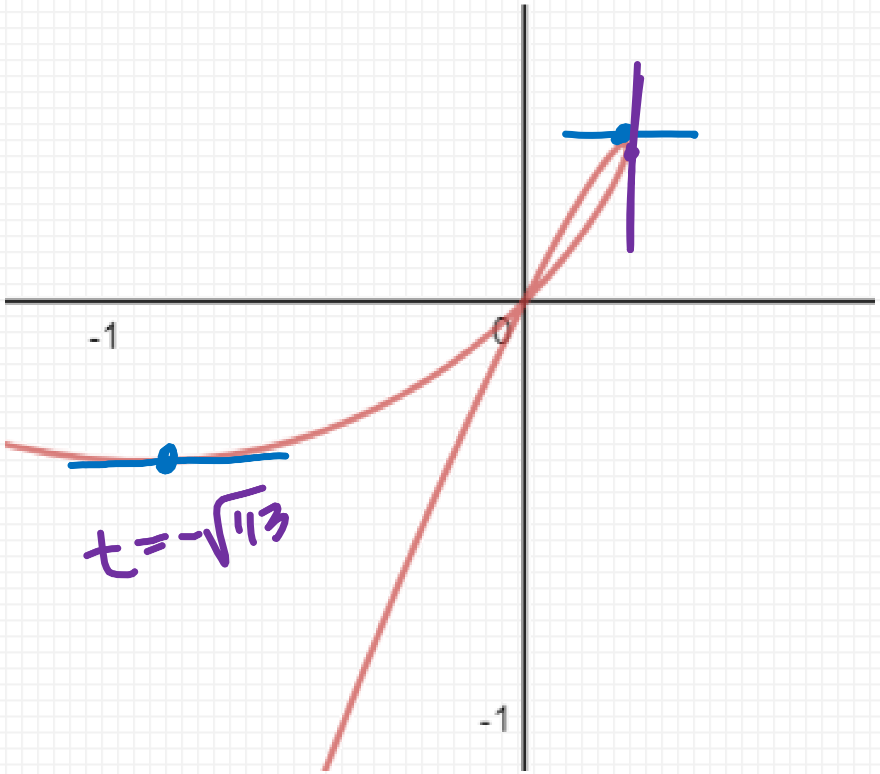
$\rightarrow$

Use the parametric equations to the right to answer the following questions.

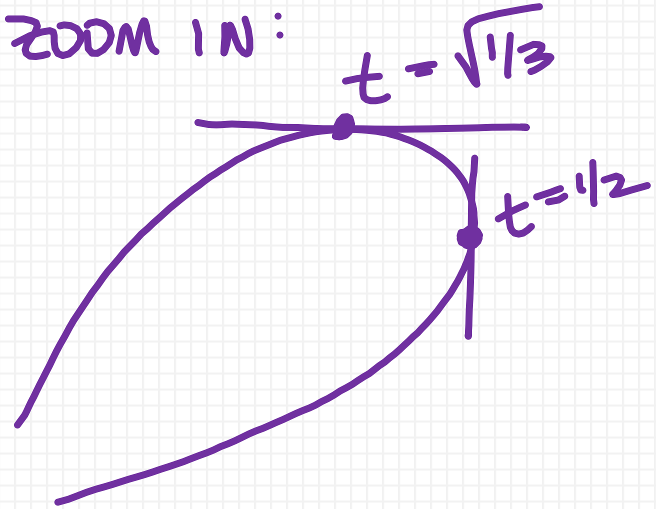
$$x = t - t^2$$

$$y = t - t^3$$

3. Find all value of *t* for which the graph has horizontal and vertical tangents.



ZOOM IN:



A prolate cycloid is given by the given parametric equations. The graph crosses itself at (0,2). Find the equations of both tangent lines at this point.

y-INTERCEPT!

$$\boxed{\begin{aligned} x &= 2t - \pi\sin t = 0 \\ y &= 2 - \pi\cos t = 2 \end{aligned}}$$

$2 - \pi\cos t = 2$

$-\pi\cos t = 0$

$\cos t = 0$

$t = \pi/2 ✓,\ -\pi/2 ✓,\ \cancel{3\pi/2},$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\pi\sin t}{2 - \pi\cos t}$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/2} = \frac{\pi\sin\frac{\pi}{2}}{2-\pi\cos\frac{\pi}{2}} = \frac{\pi}{2}$$

$$\left.\frac{dy}{dx}\right|_{t=-\pi/2} = \frac{\pi\sin\frac{-\pi}{2}}{2-\pi\cos\frac{\pi}{2}} = \frac{-\pi}{2}$$

$$\boxed{y = \pm\frac{\pi}{2}x + 2}$$

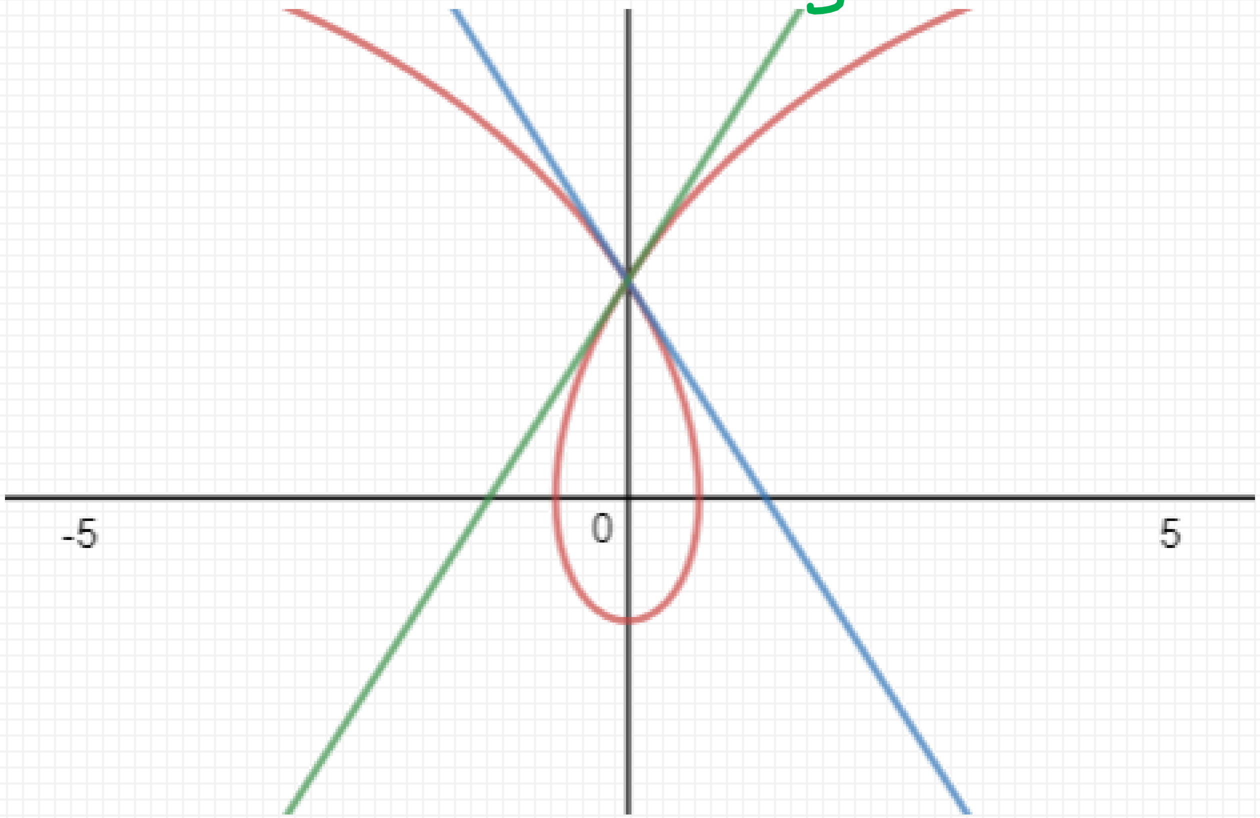
→

A prolate cycloid is given by the given parametric equations. The graph crosses itself at (0,2). Find the equations of both tangent lines at this point.

$$x = 2t - \pi \sin t$$
$$y = 2 - \pi \cos t$$

$y = -\frac{\pi}{2}x + 2$

$y = \frac{\pi}{2}x + 2$



→

Practice:

p. 518 # 4, 6

Homework:

p. 147 # 41 – 49 odd

p. 518 # 1 – 9 odd